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# A Numerical Solution to the Problem of Dynamic Indentation of an Elastic Plate by a Rigid Punch

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A numerical procedure for the solution of the elastodynamic contact problem of a half-space with a rigid indenter developed previously is generalized here to solve the dynamic contact problem for an elastic plate. Both rigidly clamped and traction-free boundary conditions on the other surface of the plate are considered. The effects of the finite thickness of the plate and the type of boundary conditions on the resulting dynamic stresses at the impacted surface of the plate are exhibited in several situations.

## INTRODUCTION

In an elastodynamic problem which involves the indentation of the surface of an elastic solid by a rigid indenter, mixed boundary conditions over a time-dependent region are encountered. As the indenter is depressed into the solid, there are some points on the surface which at first have traction-free boundary conditions, but later have displacements boundary conditions in order that the elastic body conforms itself to the shape of the indenter in the contact region. The basic difficulty in contact problems is that, in general, the time-dependent contact region is not known a priori, but must be determined from the process of the solution of the problem. A recent survey on various contact problems can be found in [1] where it can be noticed that analytical treatments to dynamic problems are known only in some simple cases.

In [2] the two-dimensional contact problem between a rigid die and an elastic half-space was treated. The method of solution given in [2] involves an iterative process which is continued until the complete solution is obtained which satisfies all the requirements of the dynamic contact problem. The method is general enough and can handle punches of arbitrary shape, as well as time-dependent cases in which the instantaneous contact velocity may be subsonic, transonic, or supersonic. This is in contradistinction to the existing analytical treatments in which the velocity of contact is assumed to be constant, and the mathematical attack of each case is different. The numerical procedure was checked with several analytical results valid in specific situations and good agreements were obtained.



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In [3] the method was generalized to the impact-contact problem of an elastic half-space impinged by an axisymmetrical projectile, and to the dynamic indentation by a rigid conical die. Here too the reliability of the method of solution was checked with an analytical solution and excellent agreement was obtained. In [2] and [3], both smooth and frictional contacts were considered.

The above treatments concerned elastic half-spaces in which the effect of the thickness of the material is ignored. In practice, however, the body possesses a finite thickness the effect of which on the dynamic field should be studied. This effect is extremely significant when the dimension of the contact region becomes comparable with the thickness of the body, and further increases with decreasing thickness.

The elastostatic field in a plate subjected to axisymmetrical rigid indentation on its upper surface, and which rests on a rigid foundation was analyzed by Tsai [4]. He found a magnification in the maximum tensile stress due to the effect of the plate thickness, and for thin plates under the pressure of large indenters, the maximum tensile stress in the plates may become twice as large as that in a half-space. This investigation was complemented in [5] by experiments which were carried out to observe the fracture produced in glass plates, impacted by spherical indenters on both surfaces of the plate, as a function of the thickness of the plate and the radius of the indenter. It turns out that for a sufficiently large indenter, the fracture load decreases rapidly as the thickness of the plate decreases, so that for a thin plate it is much less than that for a thick plate. In a more recent publication, Tsai [6] investigated the dynamic contact problem of a plate impacted by a spherical projectile. By using the integral transform technique the stresses were written as the sums of the associated static plate stresses and wave effects integrals. A practical theory was developed under the assumption of locally static response during impact, which is equivalent to keeping only the first-order term but dropping all higher-order terms obtained from contour integrations.

Recognizing the importance of the effect of thickness, we generalize in the present paper our previous methods of solution in [2, 3], to the case of the dynamic contact of a plate with a rigid indenter. The indenter impinges the plate at its upper surface, whereas its lower surface is kept either rigidly clamped or traction-free. The resulting dynamic stresses exhibit the complete effect of the finite thickness of the plate by comparing them with the extreme case of an elastic half-space. In addition, the effect of the type of the boundary conditions imposed on the lower surface of the plate is shown. Both effects are exhibited in uniform indentation and impact situations.

## Formulation of the Problem

Let us consider a homogeneous isotropic elastic plate. The elastodynamic equations of motion, in the absence of body forces, are given by

$$\rho \frac{\partial^2}{\partial t^2} \mathbf{u} = (\lambda + 2\mu) \operatorname{grad} \operatorname{div} \mathbf{u} - \mu \operatorname{rot} \operatorname{rot} \mathbf{u}, \tag{1}$$

where $\mathbf{u}$ is the displacement vector, $t$ is the time, $\lambda$ and $\mu$ are the Lamé constants of the material, and $\rho$ its density. The compressional and shear wave speeds in the solid are given respectively by $c_1 = [(\lambda + 2\mu)/\rho]^{1/2}$, $c_2 = [\mu/\rho]^{1/2}$. The plate is assumed to be initially as rest, i.e.,

$$\mathbf{u} = \frac{\partial}{\partial t}\mathbf{u} = 0 \qquad t < 0. \tag{2}$$

At time $t = 0$, a rigid punch starts to indent the plate at a velocity $V(t)$. During the indentation process, the generally unknown contact region between the punch and the plate varies with time.

We shall confine ourselves either for the two-dimensional situation in which all quantities are independent of the direction $z$ in the Cartesian coordinates $(x, y, z)$, or the axisymmetrical situation in which all quantities are independent of the polar angle $\theta$ in a cylindrical system $(r, \theta, z)$.

The boundary conditions at the loaded surface of the plate in the case of a smooth indentation are

$$\begin{aligned} &u_y = f(x, t), \qquad \sigma_{yx} = 0 \qquad &&\text{for} \quad |x| \leqslant S(t) \\ &\sigma_{yx} = \sigma_{yy} = 0 &&\text{for} \quad |x| > S(t), \quad y = 0, \quad t > 0 \end{aligned} \tag{3}$$

in the two-dimensional case in which the plate $0 \leqslant y \leqslant H$ is considered (see Fig. 1a), and it is assumed that the indenter is symmetrical about the $y$-axis. In (3), $u_i$ is the $i$th component if the displacement vector, $\sigma_{ij}$ are the components of the stress tensor, $f(x, t)$ is the prescribed vertical displacement imposed over the time-dependent region of contact at the surface and $S(t)$ describes the position of the edge of the unknown moving region, with $S(0) = 0$.

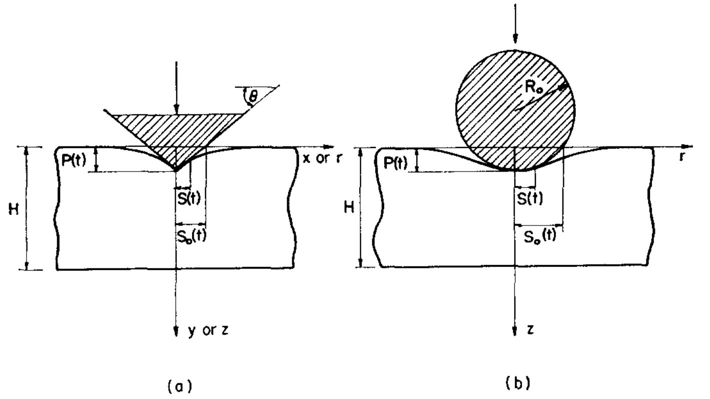


FIG. 1. (a) A wedge-shaped or conical indenter. (b) A spherical projectile.

The corresponding boundary conditions in the cylindrical coordinates appropriate to the plate $0 \leqslant z \leqslant H$ have the form

$$\begin{aligned} &u_z = f(r, t), \quad \sigma_{zr} = 0 && \text{for} \quad r \leqslant S(t) \\ &\sigma_{zz} = \sigma_{zr} = 0 && \text{for} \quad r > S(t), \quad z = 0, \quad t > 0. \end{aligned} \tag{4}$$

The other surface of the plate is chosen to be either rigidly clamped

$$\mathbf{u} = 0, \qquad y = H \text{ (or } z = H), t > 0 \tag{5}$$

or traction-free, i.e.,

$$\sigma_{yx} = \sigma_{yy} = 0, \qquad y = H, t > 0, \tag{6}$$

or

$$\sigma_{zr} = \sigma_{zz} = 0, \qquad z = H, t > 0, \tag{7}$$

where (6) and (7) correspond to the two-dimensional and three-dimensional problem with cylindrical symmetry, respectively.

To the previous conditions, the relation between the total normal force $F(t)$ between the indenter and the plate, and the rate of change of the indenter momentum should be incorporated. This relation has the form

$$m \frac{d^2}{dt^2} p(t) = F(t), \tag{8}$$

where $p(t)$ is the penetration distance of the indenter at time $t$ (i.e., the vertical displacement of the loaded boundary of the plate along the vertical axis) and $m$ represents the mass of the indenter. To (8) the initial condition of impact

$$p(0) = 0 \qquad \dot{p}(0) = V(0) \tag{9}$$

should be added.

In the case of a uniform indentation at a constant velocity $V$, $p(t) = Vt$. In the case of an axisymmetrical projectile iminging the plate, $F(t)$ is determined by the reaction force between the projectile and the plate, so that

$$F(t) = 2\pi \int_0^{S(t)} \sigma_{zz}(r, z = 0, t)\, r\, dr. \tag{10}$$

The results given in this paper concerns a rigid wedge for which $f(x, t)$ has the form (see Fig. 1a)

$$f(x, t) = p(t) - x \tan \theta. \tag{11}$$

For a conical wedge we have

$$f(r, t) = p(t) - r \tan \theta. \tag{12}$$

For a spherical projectile of radius $R_0$ (see Fig. 1b)

$$f(r, t) = p(t) - [R_0 - (R_0^2 - r^2)^{1/2}]. \tag{13}$$

The paraboloid approximation of (13) in which $R_0$ is much larger than the contact radius $S(t)$ has the form

$$f(r, t) = p(t) - r^2/2R_0. \tag{14}$$

The principle difficulty in a contact problem is the determination of the contact edge $S(t)$ from which the contact velocity $\alpha(t) = \dot{S}(t)$ can be determined.

In the following we present briefly a numerical algorithm which can solve the dynamic contact problem and satisfies all the required conditions.

## Numerical Solution

The numerical solution of the dynamic contact problems which were formulated in the previous section can be divided into two parts:

(1) In the first part explicit finite difference approximations of three level types of the equations of motion (1) are formulated in Cartesian and cylindrical coordinate systems. The explicit forms of these difference equations can be found, for example, in [7, 8] for Cartesian and cylindrical coordinates, respectively. These difference equations are of a second-order accuracy, i.e., the error resulting from the replacement of the differential equations (1) by their difference approximation is of a second-order in the increments.

(2) The second part of the numerical procedure consists of the treatment of the boundary conditions taking into account the fact that the region of contact is not known a priori, but must be determined from the requirements of the dynamic contact problem. These requirements are: (a) The normal contact stresses at the region of contact must be compressive. (b) No interpenetration outside the contact region can occur. This implies that the deformed position of the boundary of the plate outside the contact region must lie below the surface of the indenter. This condition can be expressed in the form

$$u_y > f(x, t), \qquad y = 0, t > 0 \tag{15}$$

or

$$u_z > f(r, t), \qquad z = 0, t > 0. \tag{16}$$

The numerical process for the treatment of the boundary conditions is based on the fact that if both these two requirements are satisfied as well as the relevant boundary conditions simultaneously, then the correct solution of the dynamic contact problem has been obtained. If these requirements are not satisfied then an iterative procedure, which is described in the sequel, is continued.

We start the iterations by assuming tentatively that the current position of the contact region is given by the abscissa of the point at which the rigid body intersects the surface of the plate at time $t$. For the conical wedge (12), for example, this intersection occurs at $S_0(t) = p(t)/\tan\theta$, and for the spherical projectile (14) it occurs at $S_0(t) = [2p(t)/R_0]^{1/2}$. Having determined the edge of the region of contact, we can readily impose the appropriate boundary conditions at the surfaces of the plate.

The boundary conditions at the impacted surface are imposed by introducing the explicit expressions for the stress components in the relevant coordinate system in terms of the displacement gradients, and approximating the latter by their difference expressions. Accordingly, at every time step a system of algebraic equations in the unknown surface displacements is obtained. Similarly, another system of equations are obtained for the lower plate surface. These systems are solved by the Gauss–Seidel iterative method which was found to be very effective and convenient from both rate of convergence and programming points of view (see [2] for more details). In [2] the convergence of the Gauss–Seidel iterative procedure was established.

Having computed all the surface displacements of the plate, the stresses within the assumed contact region can be calculated and verify that the previously mentioned two requirements are satisfied. If the answer is affirmative, we deduce that the correct solution at the current time $t$ has been obtained, so that we can proceed to the next time step. In the case of a negative answer, we modify the preassumed edge of contact by passing to a neighboring point and repeat the process by imposing again the boundary conditions at the impacted surface of the plate and solving the resulting system of algebraic equations for the surface displacements. This iterative process is continued until all the requirements of the contact problem as well as the boundary conditions are satisfied simultaneously yielding the correct dynamic solution.

## Applications

The numerical method is applied to solve the indentation problem of a plate by a rigid wedge penetrating at a constant speed, and to the impact–contact problem of a rigid spherical projectile with a plate.

### 1. *The Two-Dimensional Dynamic Indentation of a Plate by a Wedge*

Consider a rigid wedge-shaped die which indent a plate at a given constant speed $V$. In the special situation in which the thickness of the plate $H$ becomes very large as compared with the region of contact, we obtain the indentation problem of an elastic half-space by a rigid wedge at constant speed for which an analytical solution due to

Robinson and Thompson [9] exists. This analytical solution was obtained by employing the self-similar method of solution which is applicable to a class of elastodynamic problems that meet certain homogeneity requirements and contain no characteristic length. In the subsonic case in which the contact velocity $\alpha = \dot{S}(t)$, assumed to be constant, is less than the Rayleigh wave speed, the contact stress under the punch is given in [9]. By a numerical integration, the value of the contact velocity $\alpha$ for a given wedge angle $\theta$ and elastic constants of the half-space can be determined. The analytical solution was employed in [2] as a check to the veracity of the numerical process and excellent agreement was obtained.

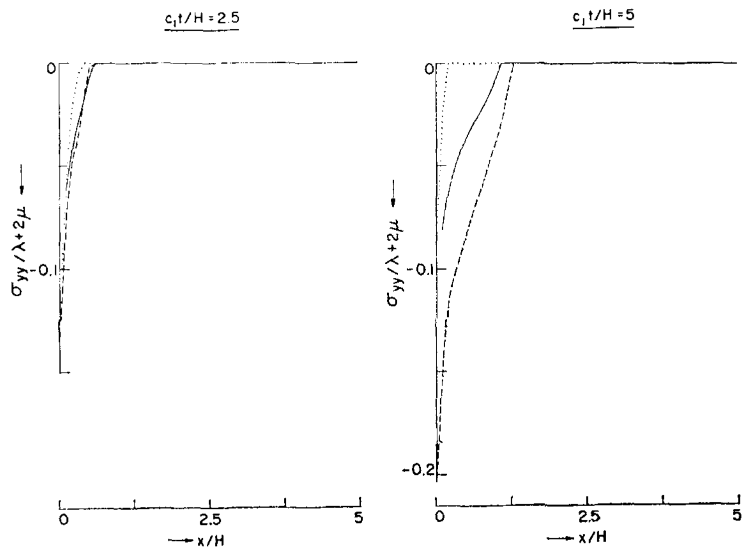


FIG. 2. Numerical solution for the contact normal stresses $\sigma_{yy}$ induced by the uniform indentation by a wedge-shaped die of a clamped (5) (dashed lines) and traction-free (6) plate (dotted lines), when $c_1t/H = 2.5$ and 5. The solid lines correspond to the analytical solution for the indentation of a half-space.

In Fig. 2 the contact stresses $\sigma_{yy}(x, 0, t)$ versus $x/H$ are given for the two types of boundary conditions (5) and (6) which correspond to a rigidly clamped and traction-free plates respectively, as well as the analytical solution for a half-space (in this case $H$ becomes a reference measure of length). The stresses are shown when $c_1t/H = 2.5$ and 5 and the indentation speed of the wedge (11) is chosen to be $V/c_2 = 0.05$ (i.e., $V/c_1 = 0.0274$) with $\tan\theta = 0.1$. The material is characterized by $(c_2/c_1)^2 = 0.3$. The effect of thickness of the plate can be clearly observed from these curves, and in addition, the type of the boundary conditions can be seen to have a significant effect on the resulting stresses. Thus, the clamped boundary conditions (5) yields more intense stresses. Furthermore, since the contact region can be easily identified in the figure, the effects of the thickness and the type of the boundary conditions of the plate on $S(t)$ can be clearly observed.

## 2. *The Axisymmetric Dynamic Indentation of a Plate by a Conical Wedge*

This is the counterpart of the previous application in the three-dimensional case with axial symmetry. Here too, an analytical solution to the dynamic indentation of an elastic half-space by the conical wedge (12) pressed at a constant speed $V$ is available [10] using the self-similar method of solution.

By a numerical integration, the value of $\alpha$ can be determined for a given indentation velocity $V$, conical wedge angle $\theta$, and elastic constants of the half-space. This analytical solution was employed in [3] as a check to the reliability of the numerical procedure and a satisfactory agreement was obtained.

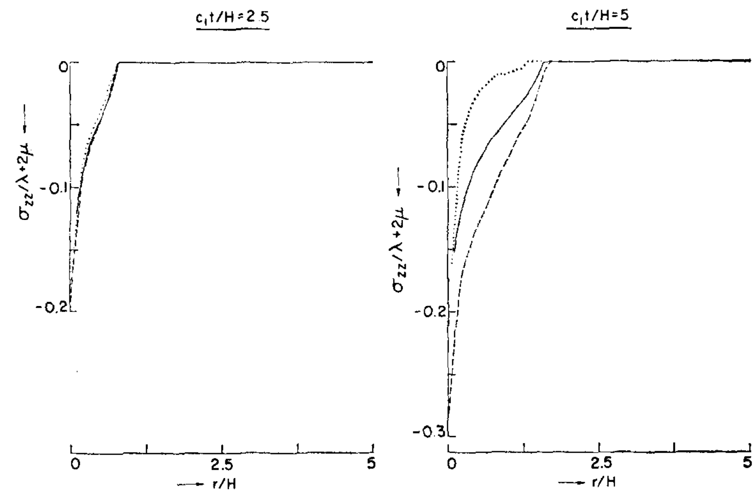


FIG. 3. Numerical solution for the contact normal stresses $\sigma_{zz}$ induced by the uniform indentation by a conical wedge of a clamped (5) (dashed lines) and traction-free (7) plate, (dotted lines), when $c_1t/H = 2.5$ and 5. The solid lines correspond to the analytical solution for the indentation of a half-space.

In Fig. 3 we exhibit the resulting contact stresses versus $r/H$ for the two types of boundary conditions (5) and (7) of the plate together with analytical solution for a half-space. The material is characterized by a Poisson ratio $\nu = \frac{1}{4}$ with the indentation velocity $V/c_2 = 0.06$ (i.e., $V/c_1 = 0.0346$) and $\tan\theta = 0.1$. Here too the effects of the finite thickness of the plate and the type of boundary conditions on the dynamic response can be easily observed.

It is interesting to compare the indentation forces which are required to press the punch at the constant speed $V$ in the various cases considered. The indentation force is given by (10). In the indentation problem of a half-space by a conical wedge at a constant speed $V$ the force is given by [10]

$$F(t) = -2\pi^2\sigma_0 t^2\alpha^2.$$

In Table I a comparison between the indentation forces in the various considered

TABLE I

The Indentation Force Required to Press the Conical Punch ($\tan\theta = 0.1$) at a Constant Speed $V/c_2 = 0.06$

| | $F(t)/(\lambda + 2\mu)H^2$ | | |
|---|---|---|---|
| $c_1t/H$ | half-space | clamped plate | traction-free plate |
| 1.25 | −0.021 | −0.0162 | −0.0006 |
| 2.5 | −0.085 | −0.083 | −0.007 |
| 3.75 | −0.190 | −0.247 | −0.087 |
| 5 | −0.337 | −0.572 | −0.073 |

situations are shown. The effect of the type of boundary conditions and the finite thickness of the plate can be noticed.

### 3. *The Impact–Contact of a Plate by an Axisymmetric Projectile*

Let us consider the dynamic contact problem of an elastic plate impacted by a rigid sphere of radius $R_0$ with an initial velocity $V$. In contrast to the previous problems of the wedges, the penetration distance $p(t)$ is not known a priori but must be computed at every time step of the solution according to (8).

In Fig. 4 the numerical solutions for the normal and radial stresses are given versus $r/H$ when $c_1t/H = 2.5$ and 5 for a large sphere (14) of radius $R_0/H = 5$, impacting

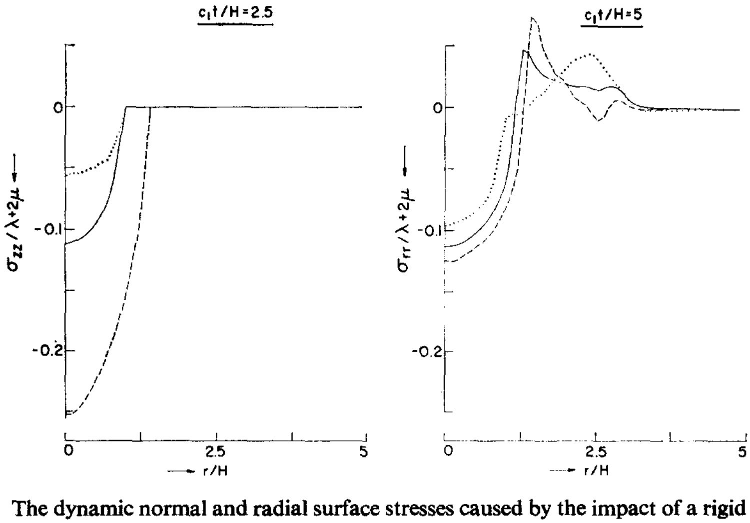


FIG. 4. The dynamic normal and radial surface stresses caused by the impact of a rigid sphere (14) with a clamped plate (5) (dashed lines) and a traction-free plate (7) (dotted lines), when $c_1t/H = 2.5$ and 5. The solid lines show the corresponding solution for an impacted half-space.

the plate at an initial velocity $V/c_1 = 0.05$. The density of the rigid sphere was arbitrary chosen to be equal to $3\rho/4\pi$, and the plate is characterized by $(c_1/c_2)^2 = 3$. Both the rigidly clamped (5) and traction-free (7) cases of a plate are shown as well as the corresponding stresses in an impacted half-space. The effects of the finite thickness of the plate and the type of boundary conditions involved on the resulting dynamic stresses and the radius of the region of contact are clearly exhibited. Special attention should be given to the radial stress $\sigma_{rr}$ since it is important in determining the possibility of the initiation and growth of a crack in the medium due to the impact. The crack might appear near the region of maximum tensile radial stress. In fact, it is well seen that the radial stresses are compressive when approaching the boundary of the region of contact from the center, whereas approaching it from outside it is tensile with a rapid change near the boundary. The plots in Fig. (4) indicate that the largest maximum tensile radial stress is obtained when the plate is rigidly clamped.

## Conclusions

A numerical procedure for the dynamic indentation and impact of an elastic plate is presented. Several solutions are given exhibiting the effect of thickness and the type of boundary conditions. The method is applicable to other unsolved problems such as the frictional indentation and impact of a plate, and the dynamic indentation and impact of an elastic anisotropic plate. Accordingly, the method presented in this paper answers to open problems presented in the review of Kalker [1, Table 3], i.e., the frictional dynamic contact problems of elastic layers.

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